# Graph Theory and Algorithms – Exercises I

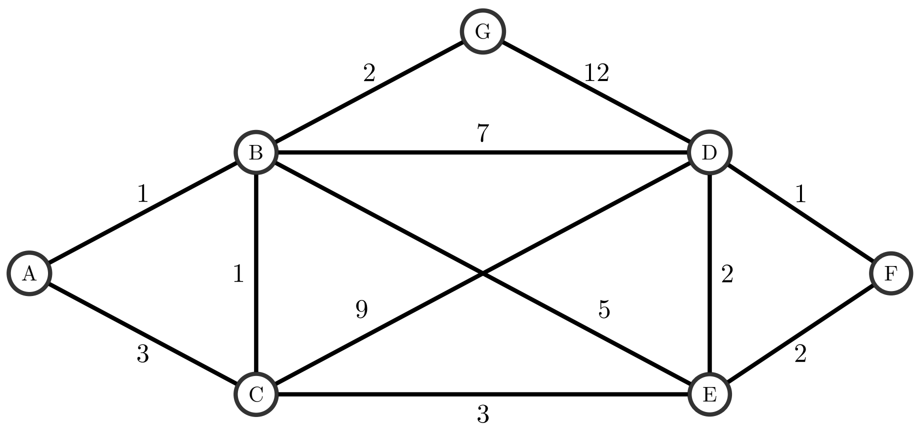
1. Prove that a graph is a tree if and only if it is connected and every edge is a cut edge (i.e. an edge whose deletion disconnects the graph).

2. Prove or disprove that every graph with fewer edges than vertices has a component that is a tree. (Note: a single vertex is a tree.)

3. Let $T$ be a tree and let $T_i = (V_i, E_i)$ be a subtree of $T$ for $i = 1, 2, 3$. Suppose that $V_i \cap V_j \neq \emptyset$ for all $1 \leq i < j \leq 3$. Prove that $V_1 \cap V_2 \cap V_3 \neq \emptyset$. Can you generalise this to more than 3 subtrees.

4. For $n \geq 3$, let $G$ be an $n$-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine the number of edges in $G$ and use this to determine $G$.

5. Apply Kruskal's algorithm to find a minimum weight spanning tree in the graph below. Write down the edges of the minimum weight spanning tree. How many minimum weight spanning trees are there?




6. Let $G = (V, E)$ be a connected graph with each edge $e$ given a positive weight $w(e)$. Recall that Kruskal's algorithm produces a minimum weight spanning tree for $G$ with weights $w$. Consider a new edge weighting $w'$ where

   (a) $w'(e) = w(e) + 37$ for all $e \in E$;

   (b) $w'(e) = 37w(e)$ for all $e \in E$;

   (c) $w'(e) = w(e)^2$ for all $e \in E$.

   In each case, if we run Kruskal's algorithm with the new weighting, will the resulting minimum weight spanning tree also be a minimum weight spanning tree with respect to the old weighting $w$? Briefly justify your answers. Which answers (if any) change if we allow $w$ to have negative weights.