Graph Theory and Algorithms – Exercises II

1. There are five cities in a network and the cost of travelling directly between city i and j is a_{ij} given by the matrix below. Determine the minimum cost of travelling between city 2 and each other city. Use Dijkstra's algorithm to determine the minimum cost of travelling between city i and j for pair of cities i and j.

$$\begin{bmatrix} 0 & 5 & 22 & 33 & 7 \\ 13 & 0 & 15 & 27 & 14 \\ \infty & 17 & 0 & 10 & 30 \\ 15 & 12 & 8 & 0 & \infty \\ 10 & 20 & \infty & 17 & 0 \end{bmatrix}$$

2. Give an example of a graph with negative edge weights in which Dijkstra's algorithm fails to find the distance between two vertices u and v when the algorithm starts at u (distances are defined as in the lectures but are now allowed to take negative values). State precisely which step in the proof of correctness of Dijkstra's algorithm fails for negative edge weights.

3. Suppose G is a connected weighted graph where all edges have distinct weights. Use the exchange property to prove that G has a unique minimum weight spanning tree.

4. Let G be a graph with a matching M that saturates $S \subseteq V$. Prove that some maximum matching also saturates S. Is it true that every maximum matching saturates S?

5. Let M and M' be matchings in a bipartite graph G with vertex classes X and Y. Suppose that M saturates $S \subseteq X$ and M' saturates $T \subseteq Y$. Prove that G has a matching that saturates $S \cup T$.

6. +Two people play a game on a graph G alternately choosing distinct vertices. Player 1 begins by choosing any vertex. Subsequently, each player on their turn must choose a vertex that has not previously been chosen and that is adjacent to the vertex chosen (by the other player) on the previous turn. Thus together the two players follow a path. The winner is the last player able to pick a vertex.

   Prove that the second player has a winning strategy if G has a perfect matching and otherwise the first player has a winning strategy. (Hint: for the second part, the first player should start with a vertex omitted by some maximum matching).

7. Let Y be a set and let $A_i \subseteq Y$ for $1 \leq i \leq m$. Prove that we can find distinct elements $a_1, \ldots, a_m \in Y$ satisfying $a_i \in A_i$ if and only if $|\cup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$. (Hint: transform this into a graph problem.)

8. Prove that a regular bipartite graph always has a perfect matching. Find an example of a regular graph of degree 3 which has no perfect matching.