Graph Theory and Algorithms – Exercises V

1. Let G be a graph with at least 4 vertices. Prove that G is 2-connected if and only if for every pair of disjoint vertex subsets $X, Y \subseteq V(G)$ satisfying $|X|, |Y| \geq 2$, we can find two completely vertex disjoint paths P_1 and P_2 which each have one endpoint in X and the other in Y but no internal vertices in $X \cup Y$.

2. Using Menger's Theorem (or otherwise) prove that if G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.

3. Use Theorem 3.10 to prove the König-Egeváry Theorem.

4. Suppose G is 2-connected but $G - e$ is not 2-connected for every $e \in E(G)$. Prove that $\delta(G) = 2$ (hint: use ear decompositions). Conclude that such a graph has at most $2n - 4$ edges where $n = |V(G)| \geq 4$.

5. Prove or disprove. If $\chi(G) = k$ then G has a k-colouring in which one of the colour classes has $\alpha(G)$ vertices.

6. Given finite sets $S_1, \ldots, S_m$, let $U = S_1 \times \cdots \times S_n$. Define a graph G with vertex set U by making u adjacent to v if and only if u and v differ on all coordinates. Determine $\chi(G)$.

7. Prove that if $\chi(G) \geq k$ then G has at least $\binom{k}{2}$ edges.

8. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n$, where $\overline{G}$ is the complement of G and n is the number of vertices of G. Use this to show that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. For each n that is a square number give an example of an n-vertex graph that achieves these bounds.