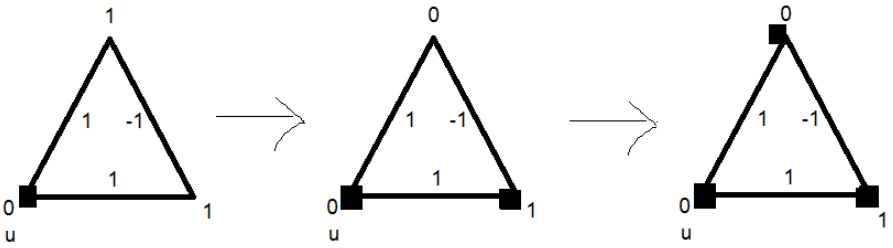
PROBLEM 1)
Dijkstra's algorithm for directed graph is like the one without directions. The only difference is that a neighbor of a vertex v is now a vertex to which one can go by an edge in the same direction as where one goes from v.
By running this algorithm we get the following matrix M, where M_{ij} is the minimal length path from i to j.

$$\begin{pmatrix} 0 & 5 & 20 & 24 & 7 \\ 13 & 0 & 15 & 25 & 14 \\ 25 & 17 & 0 & 10 & 30 \\ 15 & 12 & 8 & 0 & 22 \\ 10 & 15 & 25 & 17 & 0 \end{pmatrix}$$

PROLBEM 2)
Take the 3-cycle with edge weights $1, 1$ and -1.

Dijkstra's algorithm runs as follows:



If we call the rightmost vertex in the second step v, then something in the proof is wrong. In this case P is the edge uv and Q must be the u, v-path via the upper vertex, which must be v'. Now $w(Q) = 0$ but $w(uQv') = 1$. So the unequility is wrong.

PROBLEM 3)
Krustal's algorithms says that we can find a minimum weights spanning three, because we have a weighted connected graph. Assume that T_1 and T_2 are different MWS-trees (minimum weight spanning trees).
Let $e_1, e_2, ..., e_n$ be the edges in $T_1 - T_2$ in order from low to high weights.
Let $e'_1, e'_2, ..., e'_n$ be the edges in $T_2 - T_1$ in order from low to high weights.
If we add e'_1 to T_1, then we can remove some e_i s.t. the remaining graph is still a tree (replacement property).
It must hold that $w(e'_1) \geq w(e_i)$ because otherwise the remaining graph would have lower total weight than T_1 was, while T_1 had already a minimum total weight. And $w(e_i) \geq w(e_1)$.
In the same way it holds that $w(e_1) \geq w(e'_i) \geq w(e'_1)$. So $w(e_1) = w(e'_1)$. But $e_1 \neq e'_1$ so this is a contradiction with the fact that all edges of G have distinct weights.

PROBLEM 4)
We use observation 2.1. If M' also saturates S, we are done, because we can repeat making new matchings being the symmetric difference between the previous matching and an augmenting path. This goes on until we have no augmenting paths anymore to make. Then by theorem 2.3 we have a maximum matching, and this saturates S.

The edges from M which are not in M' are all in $E(P)$. Let xy be an edge in $E(P) \cap M$. Let p be the vertex in $E(P)$ which is the other neighbor vertex of x and let q be the other neighbor vertex of y. Then x is saturated by M', because M' has the edge px for it, and similarly it has yq to saturate y. So the fact that some edges from M are not in M' does not matter, M' saturates S.

It is not true that every maximum matching saturates S. Problem 8 states that there is a 3-regular graph with no perfect matching, i.e. a matching which does not saturate every vertex. Then there is also a maximum matching which does not saturate a vertex, say v. We can define a matching which has only one edge, with endvertex v. Now we have a counterexample. We come back to problem 8 later.

PROBLEM 5)
see 3.1.13 from http://home.ku.edu.tr/mudogan/public_html/Introduction%20to%20Graph%20Theory%202E%20-%20West.pdf

PROBLEM 7)
First we assume that all A_i have $|A_i| < \infty$.

We translate the problem into graph language as follows. We make a bipartite graph with m vertices corresponding to $1, 2, ..., m$. They together are class X. Let Y as in the problem correspond to the other class of the bipartite graph. We let this graph have the property that vertex $i \in X$ has neighborhood A_i for all i. This graph having a matching saturating X is having distinct elements as in the problem.

Let $S \in \{1, 2, ..., m\}$ be arbitrary. The neighborhood of $S \subseteq X$ is $\cup_{i \in S} A_i$. So $|\cup_{i \in S} A_i| \geq |S| \ \forall S$ iff our graph has a matching saturating X (Hall). Done for finite A_i.

Assume that some of the A_i are infinite.
For all the infinite A_i let $a_{i1}, a_{i2}, ..., a_{im} \in A_i$ be arbitrarily chosen. We define

$$A_i' = \begin{cases} A_i & \text{if } |A_i| < \infty \\ \{a_{i1}, a_{i2}, ..., a_{im}\} & \text{if } |A_i| = \infty \end{cases}$$

We do the proof of the 'iff' again, but now for possibly infinite A_i.

"$\Leftarrow$": It also holds that $|\cup_{i \in S} A_i'| \geq |S| \ \forall S$. Since all the A_i' are finite, we can find distinct $a_1, a_2, ..., a_m$ with the requirements as in the problem.

"$\Rightarrow$": We do this with a contrapositive. If $|\cup_{i \in S} A_i| < |S|$ for an S, then S can only have elements i such that A_i is finite. So $|\cup_{i \in S} A_i'| < |S|$. Then there are no elements a_i with $a_i \in A_i'$ and $i \in \{1, 2, ..., m\}$ such that they are distinct.
If we now assume that we can still find distinct elements $a_i \in A_i$ with $i \in \{1, 2, ..., m\}$, choose $\{a_{i1}, ..., a_{im}\}$ not completely arbitrarily anymore, but now such that $a_{i1} = a_i$. Then the a_i are in A_i' for all $i \in \{1, 2, ..., m\}$ and are distinct. This is a contradiction.

PROBLEM 8)
Remark: if the degree is zero, then the statement of the problem is not true.

Let $G = (V, E)$ be a regular bipartite graph with degree $d \geq 1$. Let $X, Y \subseteq V$ be the two classes of it.
We claim that $|X| = |Y|$. From the definition of a bipartite graph we know that each edge goes from X to Y, or we can also say from Y to X. So the number of edges from X to Y is equal to the number of edges from Y to X. If $|X| > |Y|$, then there are $|X|d$ edges going from X to Y, which is more than $|Y|d$, the number of edges from Y to X. This is a contradiction.

So if there is a matching saturating X, then this matching is perfect.
It suffices to show that $|N_G(S)| \geq |S|$ $\forall S \subseteq X$ (Hall).
Assume that for some S it holds that $|N_G(S)| < |S|$.
There are exactly $|S|d$ edges going from S, and they all arrive at $N_G(S)$. So there are $\geq |S|d$ edges going from $N_G(S)$. But since $|N_G(S)| < |S|$ and there are exactly $|N_G(S)|d$ going from $N_G(S)$, there are $< d|S|$ edges going from $N_G(S)$. This is a contradiction.

Example of a graph of degree 3 without perfect matching: