PROBLEM 1)
We have $\alpha(G) + \beta(G) = n$ (lemma 2.9) and $\beta(G) = \alpha'(G)$ for bipartite graphs (König-Eg'evary), so if T is bipartite then $\alpha(G) + \alpha'(G) = n$, that is $n - k = \alpha'(G)$. So it suffices to show that trees are bipartite.

Take an arbitrary vertex $s \in T$. For each other vertex x there is a unique path from s to x (prop. 1.4).
We make vertex classes X,Y as follows. If the path from s to x is even, then $x \in X$ and $x \in Y$ otherwise.

We must show that there is no edge from X to X, and no edge from Y to Y. Assume $e = pq$ is an edge from X to X. There is a unique path P of even length from s to p.
We consider two cases: $q \in V(P)$ and $q \notin V(P)$. In the first case we consider sPq. This path has even length. Since $p \notin sPq$, we have $P = sPqp$. But this path is odd, while the path from s to q must be even since $q \in X$.
So the first case cannot exist. In the second case we take $sPpq$ and this is also odd.
We now have no edges from X to X and with a similar argument no edges from Y to Y.

PROBLEM 2)
Suppose we have G an X,Y-bigraph.

"$\Rightarrow$": If G has a matching saturating X then $\forall S \subseteq X$, $|N_G(S)| \geq |S|$ by obs. 2.4

"$\Leftarrow$": First we will determine $\beta(G)$. An example of a vertex cover is X, so $\beta(G) \leq |X|$. Let $C \in V$ be a minimum vertex cover. If $C \supseteq X$, then $\beta(G)$ must be $|X|$. Otherwise, let $D \subseteq X$ be the vertices of X which are not in C. Then by assumption $|N_G(C)| \geq |C|$. All edges from C to $N_G(C)$ must have one endvertex in C, so $N_G(C) \subseteq C$. This implies that $|C| \geq |X|$ and so $\beta(G) = |X|$.

Since $\alpha'(G) = \beta(G)$ (König-E.) and $\beta(G) = |X|$, we have $\alpha'(G) = |X|$. So maximum matchings must saturate X.

PROBLEM 3)
We have $\beta(G) = \alpha'(G) < k$ (König-E.). Let C be a minimum vertex cover. One vertex of this cover cannot cover more than $\ell - 1$ edges, otherwise contradicting the numbers of edges a star can have in G. There are $< k$ vertices in C, they together cover no more than $(k-1)(\ell-1)$ edges. But they cover

all edges of G, so there are no more than $(k-1)(\ell-1)$ edges in G. And there also exists one, namely take G such that there are $k-1$ components and these components are stars with exactly $\ell-1$ edges.

PROBLEM 4)

a) If M is a maximal matching, let C be a minimum edge cover containing M. If $e \in E(C) - M$, then exactly one endpoint of e is saturated by M. Because if both endpoints are saturated by M, then we can throw e away by definition of a minimum edge cover, and if both endpoints are not saturated by M, then $M+e$ would be a matching, contradicting M being maximal.

$2|M|$ points are covered by M, and the rest of C cover the remaining $|V|-2|M|$ points. We must have $|C| = |M| + (|V| - 2|M|) = |V| - |M|$, or $|V| = |C| + |M|$. Now $|C| = \beta'(G)$ iff $|M| = \alpha'(G)$ (Gallai).

b) If C is a minimal edge cover of G, let M be a maximum matching contained in C. If $e \in E(C) - M$, then exactly one endpoint of e is saturated by M, otherwise contradicting either the definition of either M or C. The rest of the proof is like part (a).

PROBLEM 5)

a) We have to show there are no edges from $(X \cap R)$ to $(Y - R)$. Suppose $xy = e \in E(G)$ goes from $X \cap R$ to $Y - R$. Since $x \in R$, there is an M-alternating path P from a vertex $u \in U$ to x. Then P cannot contain vertices not in R, so $uPxy$ is still a path. Since U is unsaturated by M, this path starts with an edge $\notin M$. Now the almost last edge, which is the edge to x, must lie in M because uPx alternates between $X \cap R$ and $Y \cap R$. Since M is a matching, xy cannot lie in M. But then $uPxy$ must be an alternating path in M with $y \notin R$. Such a path does not exist, so we are done.

b) Suppose there is one, let it be $e = xy$ with $x \in X - R$ and $y \in Y \cap R$. Since $y \in R$, there must be an M-alternating path P from a $u \in U$ to y. Since $x \notin R$, $uPyx$ is a path. This path alternates in M. But $x \notin R$ gives a contradiction.

Since $X - R$ is saturated by M, each vertex in $X - R$ covers an edge in M. Moreover, each vertex in $X - R$ cover a different edge in M, since otherwise G was not bipartite. If also $Y \cap R$ covers different edges in M, then we are done. Namely because these edges are different with M-edges from $X - R$

by part (b) for so far. Then $|M| \geq |Q|$ holds and also $|M| \leq |Q|$ (obs. 2.6). Hence $|M| = |Q|$ would hold and by König-E. we would be done.

It suffices to show that $Y \cap R$ is saturated by M. Suppose that $y \in Y \cap R$ is not. It lies in R, so let uPy be an M-alternating path with $u \in U$. This is an M-augmenting path, contradicting M is a maximum matching.

c) Set $R = \emptyset$
If $U = \emptyset$ then stop.
$Q = (X - R) \cup (Y \cap R)$ is a minimum vertex cover.

Otherwise, for $u \in U$: take an arbitrary M-alternating path P_1 from u by making edges from u until it is not possible to add edges thereby still having an M-alternating path.

Again take an M-alternating path P_2 from u, but now as follows:
At each vertex v you come, assume uP_2v is a subpath of one of the previous paths from u already made. In this case: add an edge vx such that uP_2vx is not a subpath of previous paths if possible. Otherwise, take an arbitrary edge vx such that uP_2vx is still M-alternating. Otherwise, stop adding edges of P_2. In the other case, namely if uP_2v is not a subpath of previous paths, go on adding edges arbitrarily until it is not possible going on adding thereby still having an M-alternating path.

Go on making paths from u until you get a path which is equal to a previous path.

Take all vertices of all the paths made. Call this R. Then $Q = (X - R) \cup (Y \cap R)$ is a minimum vertex cover.

PROBLEM 6)

We find $\begin{pmatrix} 0 & 0 & 0 & 0 & 5 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{pmatrix}$ as a maximum weight transversal and $u = \begin{pmatrix} 4 & 7 & 4 & 7 & 4 \end{pmatrix}$, $v = \begin{pmatrix} 0 & 0 & 1 & 2 & 2 \end{pmatrix}$ as a minimum weight cover.

Answer to the second question: by doing the Hungarian algorithm for the matrix with entries $(max_{ij} A_{ij}) - A_{ij}$.