PROBLEM 1)
"$\Leftarrow$": We know that the maximum number of vertex disjoint X, Y paths is ≥ 2. So the smallest X, Y separator is also ≥ 2 (theorem 3.10). So by removing only one vertex we can still walk from X to Y. Since X, Y are arbitrary, G is 2-connected.

"$\Rightarrow$": We do this with a contrapositive. So we assume that $\exists\, X, Y \subseteq V$ such that there are no two vertex disjoint X, Y paths. If there are no paths from X to Y then G is not 2-connected, so we may assume that there is exactly one (vertex-disjoint) X, Y path. Then there is a vertex, say x, separating X and Y (theorem 3.10). So by removing x we already get a disconnected graph. We thus have G not 2-connected.

PROBLEM 2)
We will make use of the following theorem from
https://en.wikipedia.org/wiki/Menger%27s_theorem :
Let G be a graph and x and y two distinct vertices. Then the size of the minimum edge cut for x and y is equal to the maximum number of pairwise edge-disjoint paths from x to y.

We take x, y such that the size of the minimum edge cut for x and y is minimal. This size is $\kappa'(G)$. This is equal to the maximum number of pairwise edge-disjoint paths from x to y. These paths don't share internal vertices, because otherwise that vertex would have degree > 3. So between x and y there are $\kappa'(G)$ internally vertex disjoint paths. Then G is $\kappa'(G)$ connected. So $\kappa'(G) \leq \kappa(G)$. With Witney we conclude that $\kappa'(G) = \kappa(G)$.

PROBLEM 3)
Let G be an X, Y-bigraph. The maximum number of vertex-disjoint X, Y paths is the maximum number of vertex disjoint edges (because of considering a maximum number, it is useless to consider X, Y-paths which are not edges, because such paths have two or more edges from X to Y), which is exactly the size of a maximum matching.
If we have a smallest X, Y separator C, then each edge has one endvertex in C, so C is a vertex cover. If we have a minimum vertex cover D, then each edge (that is each X, Y-path) has one endvertex in D, so then D is an X, Y separator. So a smallest X, Y separator and a minimum vertex cover have the same cardinality.
By looking at theorem 3.10 we conclude that König Égervary is true.

PROBLEM 4)

a) A 2-connected graph has minimum degree ≥ 2 (Whitney). We know that G can be constructed by an ear-decomposition. So there is an ear last added. If $\delta(G) > 2$ then that ear must be a path of length 1, in other words an edge. Removing that edge results again in an ear-decomposition, which is 2-connected. So $\delta(G) = 2$.

b) We first consider such a graph if it has exactly 4 vertices. It can be a 4-cycle. The only thing it can also be, is a 3-cycle with an ear, and such an ear that has more than one edge. It can only have 2 edges. This information gives us the only other possible graph. It is $(1,2,3,4,(1,2),(1,4),(2,4),(2,3),(3,4))$. By removing $(2,4)$ we get a 2-connected graph. So the 4 cycle is left. This graph has the property we want it to have. It has 4 edges, which is $\leq 2*4-4=4$. So for $n=4$ it is true.

We do induction on the number of ears added.
If we don't add an ear, we only have a k-cycle with $k \geq 4$. Then $E(G) = V(G) \leq 2V(G)-4$.
Let the inequility be true for if we added m ears, where $m \in \mathbb{Z}_{\geq 0}$ some number.
Now we assume that G has $m+1$ added ears. By deleting the last added ear we have G' a graph of only m added ears.

We claim that G' still has the properties we want it to have, i.e. it is minimally 2-connected and $V(G') \geq 4$.
If there is an edge $e \in E(G')$ such that $G'-e$ is 2-connected, then $G-e$ would also be 2-connected, which is not the case. We only have to show now that G' is not a triangle. If it is, then G is a triangle with an ear added. Then by removing the right edge e we get a cycle $G-e$, which is 2-connected. Then G is not minimally 2-connected.

So we may assume that $E(G') \leq 2V(G')-4$. Let the length of the deleted ear from G be ℓ (which is ≥ 2). Then

$$\begin{aligned}
E(G) = E(G') + \ell &\leq 2V(G') - 4 + \ell \\
&= 2(V(G) - (\ell - 1)) - 4 + \ell \\
&= 2V(G) - 2\ell + 2 - 4 + \ell \\
&= 2V(G) - \ell - 2 \leq 2V(G) - 4
\end{aligned}$$

PROBLEM 5)

We take $G = (\{1,2,3,4,5,6\}, \{(1,3),(2,3),(3,4),(4,5),(4,6)\})$.

We take colorclasses $\{1,2,4\}$ and $\{3,5,6\}$. Coloring with respect to these colorclasses and given that $\chi(G) \neq 1$ gives $\chi(G) = 2$. We also have $\alpha(G) = 4$, since $\{1,2,5,6\}$ is an independent set and higher sets are not. If there is a 2-coloring in which one of the classes has $\alpha(G) = 4$ vertices, then one colorclass must be $\{1,2,5,6\}$, and then we cannot get a 2-coloring.

PROBLEM 6)
Define $i \in \{1,2,...,n\}$ s.t. $|S_i|$ is minimal. Define

$$K_j := \{(s_1, s_2, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) \in U : s_i = j\}$$

For all K_j, all pairs of elements are not adjacent. So there is a $|S_i|$-coloring. In order to get a contradiction, assume that G is $(|S_i|-1)$-colorable and that we have colored G with only $|S_i|-1$ colors. Consider

$$\{(1,1,...,1,1,),(2,2,...,2,2),...,(|S_i|,|S_i|,...,|S_i|,|S_i|)\}.$$

Two elements of them have the same color now. These two are adjacent. So this is no $(|S_i|-1)$ coloring.

PROBLEM 7)
If G is a $\chi(G)$ coloring, then there are $\binom{\chi(G)}{2}$ ways to color endpoints of an edge differently. Assume that G has less than $\binom{k}{2}$ edges. Then there are colors $i \neq j$ s.t. there are no edges with endpoints colored i,j. Then we could have chosen i being equal to j, which results in a $\chi(G)-1$ coloring. That is not true, so there are $\geq \binom{k}{2}$ edges.

PROBLEM 8)
Suppose we have a $\chi(G)$ coloring for G. Let S be a set s.t. $|S| \geq n/\chi(G)$ which has been colored with a same color. So no pairs of vertices in S are adjacent. Then in $\overline{G}$ there S does have all pairs of vertices adjacent. So must have all his vertices colored differently to get a $\chi(\overline{G})$ coloring. So $\chi(\overline{G}) \geq |S| \geq n/\chi(G)$, i.o.w. $\chi(G)\chi(\overline{G}) \geq n$.

Let $x := \chi(G)$ and $y := \chi(\overline{G})$. Then $xy \geq n = n+k$ for a $k \in \mathbb{Z}_{\geq} 0$, so $y = (n+k)/x$.
We wanna minimize $x+y = x+(n+k)/x$. Taking d/dx and equal it to zero, we get $x = \sqrt{n}$.
So $x+y = \sqrt{n}+(n+k)/\sqrt{n} \geq 2\sqrt{n}$.

Take G being $\sqrt{n}$ disjoint copies of $K_{\sqrt{n}}$. Then $\chi(G) = \sqrt{n} = \chi(\overline{G})$.