PROBLEM 1)
If we have a $\chi(G)$-coloring, take an order in which the points with color 1 come first, then the points with color 2 and so on and finally those with color $|\chi(G)|$. We will show that in the greedy algorithm a vertex in colorclass i will have a color $\leq i$. We do it by induction.

A vertex in colorclass 1 clearly does not have neighbors in that same class, and vertices outside that class have not been assigned yet. So this vertex will get color 1. Now we are done for $i = 1$.
Let the result be true for arbitrary $i < |\chi(G)|$ and let v be a vertex in class $i + 1$.
This vertex has only assigned neighbors in classes $1, 2, ..., i$. By induction those vertices all have color $\leq i$. So by greedy we color v with a color $\leq i + 1$. Now we know by induction and since the highest colorclass is colorclass $|\chi(G)|$, that all vertices have a color $\leq |\chi(G)|$. So greedy uses (at most) $|\chi(G)|$ colors for this order.

PROBLEM 2)

We start with T_1 being a graph with 2 vertices connected by an edge. The greedy algo uses 2 colors, so for $k = 1$ it's ok. If T_k is a correct tree already constructed, and say $v_1, ..., v_n$ is an ordering of it such that the greedy algo uses $k + 1$ colors, then we contruct T_{k+1} by adding leaves w_i to each vertex v_i in T_k. This tree is also correct by the following argument. If we first color all the leaves and then v_1 then v_2 then ... then v_n, the greedy algo will assign 1 to all the leaves w_i and a color for the v_i which is exactly $+1$ higher than the assigned color v_i got in T_k.

We can prove the last claim above by another induction argument. The leaves get color 1, so the vertices v_i which first got 1 will now get 2, because they all have a neighbor with color 1. We assume for certain $k \in \mathbb{N}$ that a vertex which first got color k will now get color $k + 1$. A vertex which first got color $k+1$, had neighbors with color $1, ..., k$. By the induction hypothesis its neighbors are now with color $2, ..., k + 1$. And it also has the leaves, so also a neighbor with color 1. The greedy algo says it must get color $k + 2$.

So the claim follows, and from that claim we are done.

PROBLEM 3)
a) If $C_1, C_2, ...C_p$ are the components of G, then it is sufficient to show that the component with the highest chromatic number of all the components, has chromatic number ≤ 3. Let C_i be that component.

C_i cannot be K_k for $k \geq 5$ since then $\Delta(C_i) = 4$ while $\Delta(C_i) \leq \Delta(G) \leq 3$. By assumption $C_i \neq K_4$. If it's K_k for $k = 1, 2$ or 3, then $\chi(G) \leq 3$. If it's an odd cycle, then let it be $u_1 u_2 u_2 ... u_b u_1$. If we color u_i with color 1 whenever u_i is a vertex in the cycle with i odd and $i \neq b$, and if we color u_b with color 3 and the rest with color 1, then we have a 3-coloring. So by Brooks theorem we are done.

b) For if G is only a vertex set without edges between them, then it's not true. If G has chromatic number 2 then we are done. So let's assume $\chi(G) = 3$ and let C be a component which has $\chi(C) = 3$.

We color the vertices of C with colors $1, 2, 3$. Assume wlog that color 3 has been used least often. Let $v_1, v_2, ..., v_k$ be the vertices of C colored with 3. We look at v_1. If all the neighbors of v_1 have the same color, we recolor v_1 to 1 or 2 thereby still having a proper coloring. Else if exactly one neighbor of v_1, say w_1, has another color than the other neighbors of v_1, then we delete $v_1 w_1$ and recolor v_1 yet. Since $\Delta(G) \leq 3$, only these two cases can happen. Then we do the same for v_2 and so on. Result is a 2-coloring after deleting $\leq k$ edges, which is $\leq |V(C)|$.

This can be done for all the components with chromatic number 3. The result is then a 2-colorable graph G after deleting $\leq n/3$ edges.

PROBLEM 4)
Since G'' is a supergraph of G', we have $\chi(G'') \geq \chi(G') = \chi(G) + 1$ (Mycielski theorem).

For the second statement we cosider possiblities of cliques in G''. Three types exist.

Type 1: The clique does not consist of vertices of $v_1, ..., v_n$. By construction of the u_i in G'' and because w is connected to all of the u_i, the number of vertices of such a clique is $\leq \omega(H) + 1$.

Type 2: The clique does not consist of vertices of $u_1, ..., u_n$. Then it can consist of only w or it consists only of vertices of $v_1, ... v_n$ (and not both since w is not connected to v_i $\forall i$). In either case, the number of vertices of such a clique is $\leq \omega(G)$.

Type 3: The clique contains vertices of both $v_1, ...v_n$ and $u_1, ..., u_n$. Since it is a clique and it contains vertices of $v_1, ..., v_n$, it cannot contain w since that vertex is not connected to v_i $\forall i$. Since v_i is not connected to u_i $\forall i$, it holds that this clique cannot contain all of $\{v_i, u_i\}$ given i (*). Let $v_{j_1}, v_{j_2}, ..., v_{j_m}$ and $u_{i_1}, u_{i_2}, ..., u_{i_k}$ be the vertices from $v_1, ..., v_n$ resp. $u_1, ..., u_n$ which are in this clique. By (*) we have $j_r \neq i_t$ $\forall r, t$. By definition of a clique it holds that v_{j_r} is connected to u_{i_t} $\forall r, t$. By the Mycielski construction, v_{j_r} must be connected to v_{i_t}. It also holds that the v_{i_t} are connected to each other and so do the v_{j_r} (definition clique). So we can replace all the u_{i_t} to v_{i_t} thereby still having a clique with the same number of vertices.
This clique is of type 2, so we are done.

PROBLEM 5)
We take $G[U]$ with U as in the hint. We wanna know $\mathbb{E}(V(G[U])-E(G[U]))$, because if there is a p such that this is $\geq \frac{n}{2t}$, then there is a set U so that $V(G[U])-E(G[U]) \geq \frac{n}{2t}$ and then by removing exactly one endpoint of each edge of $G[U]$, we get an independent graph of size $\geq \frac{n}{2t}$.

$$\begin{aligned}\mathbb{E}(V(G[U]) - E(G[U]) &= \mathbb{E}V(G[U]) - \mathbb{E}E(G[U]) \\ &\geq pn - p^2(nt/2)\end{aligned}$$

where the inequility is because one edge of at most $nt/2$ edges has a probability p^2 that its endpoints are both in U, i.e. that this edge is in $G[U]$. By maximizing $pn - p^2(nt/2)$ in p we find value $p = 1/t$ and filling it in we find $(1/t)n - (1/t)^2(nt/2) = \frac{n}{2t}$.

PROBLEM 6)
a) We do this by induction to $n \in \mathbb{N}$ for n-cycles.
For $n = 1, 2$ it is meaningless, but from $n = 3$ we get cycles.
If $n = 3$, we calculate

$$\begin{aligned}\chi(C_3, k) = k(k-1)(k-2) = (k-1)(k^2-2k) &= (k-1)\big((k-1)^2-1\big) \\ = (k-1)^3 - (k-1) &= (k-1)^n + (-1)^n(k-1)\end{aligned}$$

Let the result be true for some n.
Now consider C_{n+1}. Take both neighbors of an arbitrary vertex of C_{n+1}, call them v and w.
Let C' be equal to $C_{n+1} \cup vw$. We see that $wC_{n+1}vw$ is an n-cycle (taking the right way in C_{n+1}). To color C_{n+1}, two things can happen to $wC_{n+1}vw$. It can be also a proper coloring or it can be proper except for v, w having

the same color.
The first case can happen in exactly $(k-1)^n + (-1)^n(k-1)$ ways (induction hypothesis). For the second case, we first look at the proper colorings of the longest path going from v via C_{n+1} to w. This has $\chi(P_n, k) = k(k-1)^{n-1}$ ways. By a proper coloring of this path, the same two cases can happen to $wC_{n+1}vw$. So now we can calculate the number of ways for the second case:

$$\chi(P_n, k) - (k-1)^n - (-1)^n = k(k-1)^{n-1} - (k-1)^n - (-1)^n(k-1)$$

If we color like in the first case, then the last vertex of C_{n+1} can be colored in $k-2$ ways. If we color like in the second case, then the last vertex can be colored in $k-1$ ways. So we can give the following final calculation:

$$\begin{aligned}
&(k-2)\big((k-1)^n + (-1)^n(k-1)\big)\\
&+(k-1)\big(k(k-1)^{n-1} - (k-1)^n - (-1)^n(k-1)\big)\\
&= (k-2)(k-1)^n + k(k-1)^n - (k-1)^{n+1}\\
&+\big((k-2)(k-1) - (k-1)^2\big)(-1)^n\\
&= (k-1)^{n+1} - (k-1)^n + (k-1)^{n+1} + (k-1)^n - (k-1)^{n+1}\\
&+(k-1)\big((k-2) - (k-1)\big)(-1)^n\\
&= (k-1)^{n+1} + (-1)^{n+1}(k-1)
\end{aligned}$$

We did not consider $k = 0, 1$. But $\chi(C_n, 1) = 0$ since C_n is not 1-colorable, so for $k = 1$ it's also true. Same for $k = 0$.

b) If we first color the new vertex, then the other vertices can be colored with one color less. So the answer is:

$$k\chi(C_n, k-1)$$

PROBLEM 7)
We use that $\chi(G, k) = \sum_{r=1}^{n} P_r(G) \cdot k(k-1)\cdots(k-r+1)$ where P_r is as in the proof of prop. 4.1.
We only have to look at term $r = n-1$ and $r = n$ because only there we can find terms of the form ak^{n-1}, $a \in \mathbb{N}$.
The $(n-1)$-th term is $P_{n-1}(G) \cdot k(k-1)\cdots(k-n+2)$, so the coefficient for k^{n-1} is $P_{n-1}(G)$. To determine $P_{n-1}(G)$, we must count all ways to make $n-2$ independent sets consisting of singleton vertices and one independent set consisting of 2 vertices. In other words we must count all the pairs of vertices which are not connected. If G was complete, there would be $n(n-1)/2$ edges. We only have m, so $n(n-1)/2 - m$ pairs are not connected.

Now we must show that the coefficient for k^{n-1} in the n-th term is equal to $n(n-1)/2$. To do that, we first note that $P_n(G) = 1$. So we only have to show that the polynomial $k(k-1)\cdots(k-n+1)$ has coefficient $-n(n-1)/2$ for k^{n-1}. We do that with induction.
If $n = 1$ then $-n(n-1)/2 = 0$ and $k(k-1)\cdots(k-1+1) = k$ so for $n = 1$ it's true.
Let it be true for an n.
Finally we show it for $n+1$. The polynomial becomes

$$\begin{aligned} & k(k-1)\cdots(k-n+1)(k-n) \\ = & (k^n - \frac{n(n-1)}{2}k^{n-1} + ...)(k-n) \\ = & k^{n+1} - nk^n - \frac{n(n-1)}{2}k^n + ... \end{aligned}$$

so the coefficient for $k^{(n+1)-1}$ is

$$-n - n(n-1)/2 = -n + \frac{-n^2+n}{2} = \frac{-n^2-n}{2} = \frac{-n(n+1)}{2}$$

Kuratowski problem 3)
Let the cube as described be the following graph:

$$\{\{1,2,3,4,5,6,7,8\},\{(1,2),(2,3),(3,4),(4,1),(5,6),(6,7),(7,8),(8,5),(1,5),(2,6),(3,7),(4,8)\}\}$$

Then the following subgraph is in the complement of this cube: We first take the complete graph consisting of the vertices $1,3,6,7,8$.
Then we make a subdivision as follows:
edge $(7,3)$ becomes the path $(7,5,3)$
edge $(7,6)$ becomes the path $(7,4,6)$
edge $(7,8)$ becomes the path $(7,2,8)$.
By Kuratowski's theorem we are done.