mtp1

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PROBLEM 1.1a)
Let $A \in \Sigma$ be arbitrary, then Σ_A is a σ-algebra on A:

It holds that $\emptyset \in \Sigma_A$, because $\emptyset \in \Sigma$ (it is a σ-algebra) so $\emptyset = A \cap \emptyset \in \Sigma_A$.

Let $C \in \Sigma_A$. Then C can be written as $A \cap B$ for some $B \in \Sigma$.
Then $A - C = A - (A \cap B) = A - B = A \cap B^C$, and because $B^C \in \Sigma$ it holds that $A \cap B^C \in \Sigma_A$.

Let $C_k \in \Sigma_A$ for all $k \in \mathbb{N}$. These C_k can be written as $A \cap B_k$ for $B_k \in \Sigma_A$.
Then $\cup_{k\in\mathbb{N}}(A \cap B_k) = A \cap \big(\cup_k B_k\big) \in \Sigma_A$.

$\mathcal{I}_A$ is a π-system on A:
Let $A \cap B_1, A \cap B_2$ be such that $B_1, B_2 \in \mathcal{I}$, i.e. $A \cap B_1, A \cap B_2 \in \mathcal{I}_A$.
Then $(A \cap B_1) \cap (A \cap B_2) = A \cap (B_1 \cap B_2)$. This is in $\mathcal{I}_A$ because $B_1 \cap B_2 \in \mathcal{I}$ since $\mathcal{I}$ is a π-system.

PROBLEM 1.1b)
To prove: $\sigma(\mathcal{I}_A) = \Sigma_A$.

$\subseteq$:
If $\mathcal{I}_A \subseteq \Sigma_A$, then $\subseteq$ is done since Σ_A is a σ-algebra (problem 1a) and moreover a σ-algebra containing $\mathcal{I}_A$.
Let $A \cap B \in \mathcal{I}_A$. Then $B \in \mathcal{I} \subseteq \Sigma$. So $A \cap B \in \Sigma_A$.

$\supseteq$:
We have $\tilde{\Sigma}_A := \{A \cap B : B \in \tilde{\Sigma}\} = \sigma(\mathcal{I}_A)$.
The equility is true by the following:

$$\{A \cap B : B \in \tilde{\Sigma}\} = \{A \cap (E \cup F) : E \in \sigma(\mathcal{I}_A), F \in \sigma(\mathcal{I}_{S-A})\}$$
$$= \{(A \cap E) \cup (A \cap F) : E \in \sigma(\mathcal{I}_A), F \in \sigma(\mathcal{I}_{S-A})\} = \{E : E \in \sigma(\mathcal{I}_A)\} = \sigma(\mathcal{I}_A)$$

We gonna check $\tilde{\Sigma}$ is a σ-algebra on S:

$\emptyset \in \tilde{\Sigma}$ because take $E = F = \emptyset$

If $E \in \sigma(\mathcal{I}_A)$ and $F \in \sigma(\mathcal{I}_{S-A})$ then $S-(E \cup F) = (A-E) \cup ((S-A)-F) \in \tilde{\Sigma}$

If $(E_1 \cup F_1), (E_2 \cup F_2), ... \in \tilde{\Sigma}$ then $\cup_k(E_k \cup F_k) = (\cup_k(E_k)) \cup (\cup_k F_k) \in \tilde{\Sigma}$

Next we will show that $\Sigma \subseteq \tilde{\Sigma}$.
If this is true, $\Sigma_A \subseteq \tilde{\Sigma}_A = \sigma(\mathcal{I}_A)$ follows.

So we must show $\sigma(\mathcal{I}) \subseteq \tilde{\Sigma}$.
Both sides are σ-algebras and the left side is the smallest containing $\mathcal{I}$, so it suffices to know $\mathcal{I} \subseteq \tilde{\Sigma}$.
Let $I \in \mathcal{I}$. This element is

$$(I \cap A) \cup (I \cap (S - A)) \subseteq \{E \cup F : E \in \sigma(\mathcal{I}_A), F \in \sigma(\mathcal{I}_{S-A})\} = \tilde{\Sigma}$$

PROBLEM 1.2a)
$S \in \mathcal{D}_i$ for all $i \in I$, so $S \in \cap_i \mathcal{D}_i$

Let $E, F \in \cap_i \mathcal{D}_i$ with $E \subseteq F$. Then $E, F \in \mathcal{D}_\rangle$ for all $i \in I$. Since $\mathcal{D}_i$ are all d-systems, it follows that $F - E \in \mathcal{D}_i$ for all i, so $F - E \in \cap_i \mathcal{D}_i$.

Let $E_1, E_2, ... \in \cap_i \mathcal{D}_i$ with $E_k \subseteq E_{k+1}$ for all $k \in \mathbb{N}$.
The E_k all in $\mathcal{D}_i$ for all i, so $\cup_k E_k \in \mathcal{D}_i$ for all i, so $\cup_k E_k \in \cap_i \mathcal{D}_i$.

PROBLEM 1.2b)
We only check the complement property, because the others are with a same argument as part (a).
If $A \in \cap_i \Sigma_i$ then $A \in \Sigma_i \ \forall i$, hence $S - A \in \Sigma_i$, hence $S - A \in \cap \Sigma_i$.

PROBLEM 1.2c)
Since $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq d(\mathcal{C}_2)$, the answer follows.

PROBLEM 1.5a)
We will show that for arbitrary x it holds that $1_{\liminf E_n}(x) = \liminf 1_{E_n}(x)$.
If $1_{\liminf E_n}(x) = 1 \Leftrightarrow \liminf 1_{E_n}(x) = 1$
then we are done since there is only one other case for both sides of the equility of this problem, namely both zero.
We have

$$\begin{aligned} 1_{\liminf E_n}(x) = 1 &\Leftrightarrow x \in \cap_{m=1}^{\infty} \cup_{n \geq m} E_n \\ &\Leftrightarrow \exists m \text{ such that } x \in \cap_{n \geq m} E_n \end{aligned}$$

since $\liminf E_n$ is increasing. We also have

$$\begin{aligned} \liminf 1_{E_n}(x) = 1 &\Leftrightarrow \exists m \text{ such that for } n \geq m \text{ it holds that } 1_{E_n}(x) = 1 \\ &\Leftrightarrow \exists m \text{ with } x \in E_n \text{ for } n \geq m \Leftrightarrow \exists m \text{ such that } x \in \cap_{n \geq m} E_n \end{aligned}$$

PROBLEM 1.5b)
To prove: $\mu(\liminf E_n) \leq \liminf \mu(E_n)$

On the left side we have

$$\liminf E_n = \cup_{m=1}^{\infty} \cap_{n\geq m} E_n = \lim_{k\to\infty} \cup_{m=1}^{k} \cap_{n\geq m} E_n$$

so

$$\mu(\liminf E_n) = \mu(\lim_{k\to\infty} \cup_{m=1}^{k} \cap_{n\geq m} E_n)$$
$$= \lim_{k\to\infty} \mu(\cup_{m=1}^{k} \cap_{n\geq m} E_n) = \lim_{k\to\infty} \mu(\cap_{n\geq k} E_n)$$

Now we gonna compare $\lim_{k\to\infty} \mu(\cap_{n\geq k} E_n)$ with the right side of the equility of this problem, $\lim_{k\to\infty} \inf_{n\geq k} \mu(E_n)$.
If $k \in \mathbb{N}$ is arbitrary, then $\mu(\cap_{n\geq k} E_n) \leq \mu(E_i)$ for all $i \geq k$.
So $\mu(\cap_{n\geq k} E_n) \leq \inf_{n\geq k} \mu(E_n)$.
With this, the equility of this problem is true.

PROBLEM 1.9)
$\mathcal{A}$ is an algebra:

$\emptyset \in \mathcal{A}$ because $|\emptyset| = 0 < \infty$.

If $A \in \mathcal{A}$ then by definition of $\mathcal{A}$ also $\Omega - A \in \mathcal{A}$.

If $A, B \in \mathcal{A}$, then if $|A|, |B| < \infty$ also $A \cup B \in \mathcal{A}$.
Else if wlog $|A| = |\mathbb{N}|$ then $|\Omega - (A \cup B)| \leq |\Omega - A| < \infty$.

If $\mathcal{A}$ is a π-system, then $d(\mathcal{A}) = \sigma(\mathcal{A})$ would hold (lemma 1.14).
If $A, B \in \mathcal{A}$ and $|A|$ or $|B|$ is $< \infty$, then $|A \cap B| < \infty$ so $A \cup B \in \mathcal{A}$.
Else if wlog $|A|, |B| = |\mathbb{N}|$ then $|\Omega - (A \cap B)| = |(\Omega - A) \cup (\Omega - B)| < \infty$.
So now we have a π-system and it suffices to determine $\sigma(\mathcal{A})$.
Claim: $\sigma(\mathcal{A}) = \{A \subseteq \Omega : |A| \leq |\mathbb{N}| \vee |\Omega - A| \leq |\mathbb{N}|\} =: \mathcal{A}'$.

$\subseteq$:
It is clear that $\mathcal{A} \subseteq \mathcal{A}'$.
We check $\mathcal{A}'$ is a σ-algebra:

$\emptyset \in \mathcal{A}'$ and $A \in \mathcal{A}' \Rightarrow \Omega - A \in \mathcal{A}'$ in the same way as we did before at $\mathcal{A}$ to check it is an algebra

If $A_1, A_2, ... \in \mathcal{A}'$ and if they are all countable, then $\cup A_i$ is also countable.

Else if some A_i is not countable, then $\Omega - A_i$ is, so $|\Omega - \cup A_i| \leq |\Omega - A_i| \leq |\mathbb{N}|$.

$\supseteq$:
Let $A \in \mathcal{A}'$. If this is countable, it is a countable union of singletons, so then $A \in \sigma(\mathcal{A})$.
Else if $\Omega - A$ is countable, then $\Omega - A \in \sigma(\mathcal{A})$ so $A \in \sigma(\mathcal{A})$.

PROBLEM 1.12)
We can do it with the Cantor set, but there is s.t. simpler.

Assume that $\mathcal{A}$ is an algebra. Then the complement rule $A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$ is valid.

Now we gonna show that $I_1, I_2, ... \in \mathcal{I}(\mathbb{R}) \Rightarrow \cap I_i \in \mathcal{A}$. Suppose we have a sequence $(I_n)_n$ in $\mathcal{I}(\mathbb{R})$. All these sequence elements are also in $\mathcal{A}$ (by taking $I_n \cup \emptyset \cup \emptyset \cup ...$), so for all n we also have I_n^C in $\mathcal{A}$. So $I_n^C = \cup_{i=1}^{\infty} I_{n,i}$ with $I_{n,i} \in \mathcal{I}(\mathbb{R})$.
Then $\cup_{n \in \mathbb{N}} I_n^C = \cup_{n,i \in \mathbb{N} \times \mathbb{N}} I_{n,i}$ is a countable union of elements in $\mathcal{I}(\mathbb{R})$, so $\cup_n I_n^C \in \mathcal{A}$. Again by taking the complement rule, $\cap_n I_n = (\cup_n I_n^C)^C \in \mathcal{A}$.

We can apply this for the closed intervals $[0, 1/n]$ with $n \in \mathbb{N}$.
Of course these are all in $\mathcal{I}(\mathbb{R})$. Then $\{0\} = \cap_{n=1}^{\infty} [0, 1/n] \in \mathcal{A}$. But by definition of $\mathcal{A}$, elements of $\mathcal{A}$ are always empty or are a union of intervals, which implies they are empty or have a subset being an interval. Then there exist no elements of $\mathcal{A}$ being singletons.