mtp4

Rocco van Vreumingen

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PROBLEM 4.8)
For $f = 1_A$ and $A \in \Sigma$, we get

$$\mu(1_E f) = \mu(1_{E \cap A}) = \mu(E \cap A) = \mu_E(E \cap A) = \mu_E(1_{A \cap E}) = \mu_E(f_E)$$

Now $1_E f \in \mathcal{L}^1(\mu) \Leftrightarrow \infty > \mu(1_E f) = \mu_E(f_E) \Leftrightarrow f_E \in \mathcal{L}^1(\mu_E)$.

Let $f = \sum_{i=1}^{n} c_i 1_{A_i}$ with $c_i \geq 0$. Then

$$\mu(1_E f) = \mu(1_E \sum_i c_i 1_{A_i}) = \mu(\sum_i c_i 1_{A_i \cap E}) = \sum_i c_i \mu(1_{A_i \cap E})$$
$$= \sum_i c_i \mu_E(1_{A_i \cap E}) = \mu_E(\sum_i c_i 1_{A_i \cap E}) = \mu_E(f_E)$$

Let $f \geq 0$ and $(f_n)_n \to f$ an increasing sequence of simple functions. Then

$$\mu(1_E f) = \mu(\lim_n 1_E f_n) = \lim_n \mu(1_E f_n) = \lim_n \mu_E((f_n)_E) = \mu_E(f_E)$$

Finally let $f \in \Sigma$ be arbitrary. Then

$$\mu(1_E f) = \mu(1_E(f^+ - f^-)) = \mu(1_E f^+ - 1_E f^-)$$
$$= \mu(1_E f^+) - \mu(1_E f^-) = \mu_E((f^+)_E) - \mu_E((f^-)_E) = \mu_E(f_E)$$

We use this derivation, which only holds in right conditions, in the following way:
$1_E f \in \mathcal{L}^1(\mu) \Leftrightarrow \mu_E(f_E) = \mu(1_E f) \in \mathbb{R} \Leftrightarrow f_E \in \mathcal{L}^1(\mu_E)$

PROBLEM 4.9)
ν is a function from Σ to $[0, \infty]$ (because $f \geq 0$) with the following properties:

$\nu(\emptyset) = \int_\emptyset f d\mu = 0$

If $E_1, E_2, ... \in \Sigma$ are pairwise disjoint, then

$$\nu(\cup E_i) = \int_{\cup E_i} f d\mu = \int f \sum_i 1_{E_i} d\mu = \int \sum_i f 1_{E_i} d\mu$$
$$= \int \lim_{n\to\infty} \sum_{i=1}^n f 1_{E_i} d\mu = \lim_{n\to\infty} \int \sum_{i=1}^n f 1_{E_i} d\mu \text{ (MCT)}$$
$$= \lim_{n\to\infty} \sum_{i=1}^n \int f 1_{E_i} d\mu = \sum_{i=1}^\infty \int f 1_{E_i} d\mu = \sum_i \nu(E_i)$$

PROBLEM 4.10)
Let $E \in \Sigma$ be arbitrary. Then by definition of ν, if $h = 1_E$ then

$$\nu(h) = \nu(1_E) = \nu(E) = \int_E f d\mu = \mu(f 1_E) = \mu(fh)$$

and so $h \in \mathcal{L}^1(\nu) \Leftrightarrow \infty > \nu(h) = \mu(fh) \Leftrightarrow fh \in \mathcal{L}^1(\mu)$.

Let $h = \sum_{i=1}^n c_i 1_{E_i}$ with $c_i \geq 0$. Then

$$\nu(h) = \nu(\sum_i c_i 1_{E_i}) = \sum_i c_i \nu(E_i)$$
$$= \sum_i c_i \mu(1_{E_i} f) = \mu(\sum_i c_i 1_{E_i} f) = \mu(hf)$$

and so $h \in \mathcal{L}^1(\nu) \Leftrightarrow \infty > \nu(h) = \mu(fh) \Leftrightarrow fh \in \mathcal{L}^1(\mu)$.

Let $h \in \Sigma^+$. If $h = \lim_n h_n$ for increasing simple functions $h_n \in \Sigma^+$, then

$$\nu(h) = \lim_n \nu(h_n) = \lim_n \mu(h_n f) = \mu(hf) \text{ } (f \geq 0 \text{ so MConv Thm applies})$$

and so again $h \in \mathcal{L}^1(\nu) \Leftrightarrow \infty > \nu(h) = \mu(fh) \Leftrightarrow fh \in \mathcal{L}^1(\mu)$.

Let $h \in \Sigma$. Then

$$h \in \mathcal{L}^1(\nu) \Leftrightarrow \mathbb{R} \ni \nu(h) = \nu(h^+) - \nu(h^-)$$
$$= \mu(fh^+) - \mu(fh^-) = \mu(fh) \Leftrightarrow fh \in \mathcal{L}^1(\mu)$$

PROBLEM 4.11)
Let $h = 1_A$ for $A \in \mathcal{F}$. Then

$$\mathbb{E}h(X) = \mathbb{E}1_{\{X \in A\}} = P(X \in A)$$
$$= P^X(A) = P^X(1_A) = \int 1_A dP^X = \int h dP^X$$

and so for this h it holds that

$$h \circ X \in \mathcal{L}^1(\Omega, \mathcal{F}, P) \Leftrightarrow \infty > \mathbb{E}h(X) = \int h dP^X \Leftrightarrow h \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, P^X)$$

Let $h = \sum_{i=1}^n c_i 1_{A_i}$ with $c_i \geq 0$. Then

$$\mathbb{E}h(X) = \mathbb{E}\sum_{i=1}^n c_i 1_{A_i}(X) = \sum_i c_i \mathbb{E}1_{A_i}(X)$$
$$= \sum_i c_i \int 1_{A_i} dP^X = \int \sum_i c_i 1_{A_i} dP^X = \int h dP^X$$

and so for this h it holds also.
Let $h \geq 0$. This is equal to $\lim_n h_n$ with $(h_n)_n \in \mathcal{B}^+$ an increasing sequence of simple functions. Then

$$\mathbb{E}h(X) = \lim_n \mathbb{E}h_n(X) \text{ (MCT) } = \lim_n \int h_n dP^X = \int h dP^X$$

and so for this h is holds also.

Let h be given arbitrary. Then

$$h \circ X \in \mathcal{L}^1(\Omega, \mathcal{F}, P) \Leftrightarrow \int h dP^X = \int h^+ dP^X - \int h^- dP^X$$
$$= \mathbb{E}h^+(X) - \mathbb{E}h^-(X) = \mathbb{E}h(X) \in \mathbb{R} \Leftrightarrow h \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, P^X)$$

Maybe the equality $\mathbb{E}h^+(X) - \mathbb{E}h^-(X) = \mathbb{E}h(X)$ is not clear.
We will show $h^+(X) = h(X)^+$:
Let $\omega \in \Omega$ be arbitrary. Suppose $X(\omega) = y$. Then

$$h^+(X)(\omega) = h^+(y) = \max\{0, h(y)\}$$
$$h(X)^+(\omega) = \max\{0, h(X)(\omega)\} = \max\{0, h(y)\}$$

PROBLEM 4.19a)
We calculate the following:

$$|f|_p = \mu\Big(|f|^p\Big)^{1/p} \leq \mu\Big((|f|_\infty 1_{\{f\neq 0\}})^p\Big)^{1/p} \leq \mu\Big((|f|_\infty)^p 1_{\{f\neq 0\}}\Big)^{1/p}$$
$$= \Big((|f|_\infty)^p \mu(f\neq 0)\Big)^{1/p} = |f|_\infty \cdot \mu(f\neq 0)^{1/p} \xrightarrow{p\to\infty} |f|_\infty \text{ since } \mu(f\neq 0)<\infty$$

So

$$\limsup_{p\to\infty} |f|_p \leq \limsup_{p\to\infty} |f|_\infty \cdot \mu(f\neq 0)^{1/p} = \lim |f|_\infty \cdot \mu(f\neq 0)^{1/p} = |f|_\infty$$

PROBLEM 4.19b)
Let $\epsilon > 0$ be such that $\epsilon < |f|_\infty$. First we calculate that

$$\mu(|f|^p)^{1/p} \geq \mu\Big(((|f|_\infty - \epsilon)1_{\{|f|\geq |f|_\infty-\epsilon\}})^p\Big)^{1/p} = \mu\Big((|f|_\infty-\epsilon)^p 1_{\{|f|\geq |f|_\infty-\epsilon\}}\Big)^{1/p}$$
$$= \Big((|f|_\infty-\epsilon)^p \mu(|f|\geq |f|_\infty-\epsilon)\Big)^{1/p} = (|f|_\infty-\epsilon)\mu(|f|\geq |f|_\infty-\epsilon)^{1/p} \xrightarrow{p\to\infty} |f|_\infty-\epsilon$$

where $\mu(|f|\geq |f|_\infty-\epsilon)>0$ by definition of $|.|_\infty$.

So $\liminf |f|_p \geq |f|_\infty - \epsilon$. Since $\epsilon > 0$ could have been as small as we wanted, we conclude that $\liminf |f|_p \geq |f|_\infty$.

PROBLEM 4.19c)
We will show that $|X|_{p+1}/|X|_p \geq 1$:

$$\frac{|X|_{p+1}}{|X|_p} = \frac{(\mathbb{E}|X|^{p+1})^{1/(p+1)}}{(\mathbb{E}|X|^p)^{1/p}} \geq 1 \Leftrightarrow \frac{\mathbb{E}|X|^{p+1}}{(\mathbb{E}|X|^p)^{(p+1)/p}} \geq 1$$

and the most right side is true because of the denominator as follows:

$$(\mathbb{E}|X|^p)^{(p+1)/p} \leq \mathbb{E}(|X|^{p+1})$$

since $x\mapsto x^{(p+1)/p}$ is convex on $[0,\infty)$ (Jensen).