mtp2

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PROBLEM 5.2)
Let $f \in \mathcal{L}^1(\mu)$.
We know that $\mu(f^+) = \mu_2(I_2^{f^+})$ and $\mu(f^-) = \mu_2(I_2^{f^-})$.
So

$$\mu(f) = \underbrace{\mu(f^+)}_{<\infty} - \underbrace{\mu(f^-)}_{<\infty} = \mu_2(I_2^{f^+}) - \mu_2(I_2^{f^-})$$

$$= \mu_2\left(\int f^+(s_1,(\cdot))d\mu_1(s_1)\right) - \mu_2\left(\int f^-(s_1,(\cdot))d\mu_1(s_1)\right)$$

$$= \mu_2\left(\int f^+(s_1,(\cdot))d\mu_1(s_1) - \int f^-(s_1,(\cdot))d\mu_1(s_1)\right)$$

$$= \mu_2\left(\int f^+(s_1,(\cdot)) - f^-(s_1,(\cdot))d\mu_1(s_1)\right)$$

$$= \mu_2\left(\int f(s_1,(\cdot))d\mu_1(s_1)\right) = \mu_2(I_2^f)$$

Similarly, $\mu(f) = \mu_1(I_1^f)$ and thus $(5ii)$ also holds for $f \in \mathcal{L}^1$.

To explain why $f((\cdot), s_2)$ is in $\mathcal{L}^1(\Sigma_1)$, firstly the Σ_1-measurability follows from prop. 5.1.
Secondly we must show $\mu_1\left(f^{\pm}((\cdot), s_2)\right) < \infty$ for all s_2 a.e.
I.o.w. we must show $I_2^{f^{\pm}}(s_2) < \infty$ for all s_2 a.e.
Suppose that $I_2^{f^+}(s_2) = \infty$ for all $s_2 \in A$, for some A with $\mu_2(A) > 0$.
Then $\mu(f^+) = \mu_2(I_2^{f^+}) \geq \infty \cdot \mu(A) = \infty$, a contradiction to the fact that $f \in \mathcal{L}^1(\mu)$.
Similarly it holds that $I_2^{f^-}(s_2) < \infty$ for all s_2 a.e. Let's say that this holds on $S_2 - N_a$ with N_a a set of measure zero, and that $I_2^{f^+}(s_2) < \infty$ holds on $S_2 - N_b$ with N_b a set of measure zero. Then we conclude that $s_1 \mapsto f(s_1, s_2)$ is in $\mathcal{L}^1(\mu_1)$ on $S_2 - (N_a \cup N_b)$, so integrable μ_2-a.e.

We have that

$$I_2^f = \int f(., s_2)d\mu_1 = \int f^+(., s_2)d\mu_1 - \int f^-(., s_2)d\mu_2 = \underbrace{I_2^{f^+}}_{<\infty \text{ on } N_b^c} - \underbrace{I_2^{f^-}}_{<\infty \text{ on } N_a^c}$$

So I_2^f exists on $S_2 - (N_a \cup N_b)$.

PROBLEM 5.3)

$\Rightarrow$:

To check that X_1, X_2 are random variables, let $B_1, B_2 \in \mathcal{B}(\mathbb{R})$.
Then $B_1 \times \mathbb{R} \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$ so
$\{X_1 \in B_1\} = \{X_1 \in B_1\} \cap \underbrace{\{X_2 \in \mathbb{R}\}}_{=\Omega} = \{(X_1, X_2) \in B_1 \times \mathbb{R}\}$ is measurable, and similarly the same holds for $\{X_2 \in B_2\}$. So now X_1, X_2 are measurable as well.

$\Leftarrow$:

Let again $B_1, B_2 \in \mathcal{B}(\mathbb{R})$. Then $\{(X_1, X_2) \in B_1 \times B_2\} = \{X_1 \in B_1\} \cap \{X_2 \in B_2\}$ is measurable. Since $B_1 \times B_2$ was an arbitrary element of the set which generates $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ and this is the same as $\mathcal{B}(\mathbb{R}^2)$, then this is enough to conclude that (X_1, X_2) is a random vector.

PROBLEM 5.4)

Proof for the first claim:

$\Rightarrow$:

Let $B_1, B_2 \in \mathcal{B}(\mathbb{R})$. Then

$$P^{(X_1,X_2)}(B_1 \times B_2) = P(X_1 \in B_1, X_2 \in B_2)$$
$$= P(X_1 \in B_1)P(X_2 \in B_2) = P^{X_1} \times P^{X_2}(B_1 \times B_2)$$

So the function $P^{(X_2,X_2)}$ is equal to $P^{X_1} \times P^{X_2}$ for a set which generates $\mathcal{B}(\mathbb{R}^2)$. This generator is a π-system, it also holds that these functions coincide on $\mathbb{R}^2$, the value of it is $1 < \infty$. So these functions coïncide on $\mathcal{B}(\mathbb{R}^2)$ (thm 1.16).

$\Leftarrow$:

We must show $P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1)P(X_2 \in B_2)$.
The left side equals $P^{(X_1,X_2)}(B_1 \times B_2)$, the right side equals
$P^{X_1} \times P^{X_2}(B_1 \times B_2)$. So by assumption this is true.

The proof for the second claim ("This in turn happens...") follows from cor. 3.13.

Proof for the third claim:

$\Rightarrow$:

Since X_1, X_2 are independent, then

$$\int\int_{(-\infty,x_1]\times(-\infty,x_2]} f(x,y)dxdy = P^{(X_1,X_2)}\Big((-\infty,x_1]\times(-\infty,x_2]\Big)$$
$$= P^{X_1}\Big((-\infty,x_1]\Big)P^{X_2}\Big((-\infty,x_2]\Big)$$
$$\int_{(-\infty,x_1]} f_{X_1}(x)dx \int_{(-\infty,x_2]} f_{X_2}(y)dy \text{ (problem 5.5)}$$
$$= \int_{(-\infty,x_1]}\int_{(-\infty,x_2]} f_{X_2}(y)dy f_{X_1}(x)dx$$
$$= \int_{(-\infty,x_1]}\int_{(-\infty,x_2]} f_{X_1}(x)f_{X_2}(y)dydx$$
$$= \int\int_{(-\infty,x_1]\times(-\infty,x_2]} f_{X_1}(x)f_{X_2}(y)dxdy \text{ (cuz } \lambda\text{-measure is } \sigma\text{-finite, Fubini)}$$

Now we gonna use the measures $\int\int_{(\cdot)} f(x,y)dxdy$ and $\int\int_{(\cdot)} f_{X_1}(x)f_{X_2}(y)dxdy$. We have just shown enough to conclude that these are the same measures (thm 1.16).
So $\int\int_{f(x,y)\geq f(x)f(y)} f(x,y)-f(x)f(y)dxdy = 0$. Then $f(x,y)-f(x)f(y) = 0$ a.e. (lemma 4.11).

$\Leftarrow$:
The equalities in the last display hold, but now because the first and last expression are equal. Then the second equality is by conclusion true, so X_1 and X_2 are independent (cor. 3.13).

PROBLEM 5.5)
Note that the term *density* is also explained in the theory text. It is therefore non-negative and measurable.
We have $P^X(B) = P^{(X,Y)}(B\times\mathbb{R}) = \int\int_{B\times\mathbb{R}} f(x,y)dxdy$.

By Fubini the final step to the proof is done.

PROBLEM 5.7a)
We must prove that $\mathbb{E}\sum_{k=1}^{\infty} Z_k = \sum_{k=1}^{\infty}\mathbb{E}Z_k$.
We call τ the counting measure on $\mathbb{N}$, so that this is the same as to prove

$$\int\int Z_k d\tau(k)dP = \int\int Z_k dP d\tau$$

So all we have to do is showing that Fubini holds. We know that P and τ are σ-finite, so we are done if we show $(\omega,k)\mapsto Z_k(\omega)$ is measurable (since

the Z_k are non-negative rv's).
Take $B \in \mathcal{B}$ arbitrary. Then $\{Z_k(\omega) \in B\} = \cup_{k \in \mathbb{N}} \{Z_k \in B\}$, which is of course measurable.

If $\sum_{k=1}^{\infty} \mathbb{E} Z_k < \infty$, then we have just shown

$$\mathbb{E} \sum_{k=1}^{\infty} Z_k = \sum_{k=1}^{\infty} \mathbb{E} Z_k < \infty$$

If now $P(\sum_{k=1}^{\infty} Z_k = \infty) > 0$ then $\mathbb{E} \sum_{k=1}^{\infty} Z_k = \infty$, contradiction. So this probability is 0.

PROBLEM 5.11)
Here by λ we denote the Lebesgue measure restricted to $[0,\infty)$.
The following holds:

$$E(X^\alpha) = \int\int_0^X \alpha y^{\alpha-1} dy dP = \int\int \alpha y^{\alpha-1} 1_{[0,X)}(y) d\lambda(y) dP = ...$$

If Fubini holds, we would get

$$\begin{aligned} ... = \int\int \alpha y^{\alpha-1} 1_{[0,X)}(y) dP d\lambda(y) &= \int \alpha y^{\alpha-1} \int 1_{[0,X)}(y) dP d\lambda(y) \\ = \int \alpha y^{\alpha-1} P(X > y) d\lambda(y) &= \int \alpha y^{\alpha-1}(1 - F(y)) d\lambda(y) \\ &= \alpha \int_0^\infty x^{\alpha-1}(1 - F(x)) dx \end{aligned}$$

We have that the measures P and λ are σ-finite.
Further, $(y,\omega) \mapsto \alpha y^{\alpha-1} 1_{[0,X(\omega))}(y)$ is ≥ 0. So Fubini holds if $(y,\omega) \mapsto \alpha y^{\alpha-1} 1_{[0,X(\omega))}(y)$ is measurable. So that is the last thing we must show.

For this, we are done if we show that $(y,\omega) \mapsto 1_{[0,X(\omega))}(y)$ is measurable.

Step 1)
$f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : (x,y) \mapsto 1_{[0,x)}(y)$ is $\mathcal{B}(\mathbb{R}^2)$-measurable.
For this it suffices to show $f^{-1}\{1\}$ is measurable.
This is $\{y < x\}$, which corresponds to the area under the graph of $f(x) = x$.
We can take a limit of f by taking an increasing sequence $(f_n)_n$ of simple functions with $f_n < f$ as in example 4.13. The corresponding areas under the graphs of $(f_n)_n$ are $\mathcal{B}(\mathbb{R}^2)$-measurable, and hence also the area under f (continuity property of measures).

Step 2)
$g : (\omega, y) \mapsto (X(\omega), y)$ is measurable.
If $B_1, B_2 \in \mathcal{B}(\mathbb{R})$, then $g^{-1}(B_1 \times B_2) = \{X \in B_1\} \times \{y \in B_2\}$.

Step 3)
$(\omega, y) \mapsto 1_{[0,X(\omega))}(y)$ is equal to $f \circ g$, so measurable.