# mtp7

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PROBLEM 8.1)

Choose $\alpha_i = \begin{cases} \int_{A_i} XdP/P(A_i) \text{ if } P(A_i) > 0 \\ 0 \text{ otherwise} \end{cases}$. Then we get for arbitrary element in $\sigma(\mathcal{G})$, that is for $\cup_{i\in N} A_i$ with $N \subseteq \mathbb{N}$,

$$\begin{aligned} \int_{\cup_N A_i} \sum_{i=1}^{n} \alpha_i 1_{A_i} dP = \sum_{i\in N} \alpha_i P(A_i) &= \sum_{i\in N, P(A_i)>0} \alpha_i P(A_i) \\ &= \sum_{i\in N, P(A_i)>0} \int_{A_i} XdP = \int_{\cup_N A_i} XdP \end{aligned}$$

PROBLEM 8.3a)

Let $B \in \mathcal{B}(\mathbb{R})$. Then $\{X \in B\} \in \mathcal{B}[-1,1]$ because $X$ is continuous hence measurable. Further it holds that

$$\omega \in \{X \in B\} \Leftrightarrow X(\omega) \in B \Leftrightarrow X(-\omega) \in B \Leftrightarrow -\omega \in \{X \in B\}$$

so $\{X \in B\}$ is in $\{E \in \mathcal{B}([-1,1]) : E = -E\}$.

Conversely, let $E = -E$ be an element in $\mathcal{B}([-1,1])$. We want it to be in $\sigma(X)$.
For this, we look at the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R} : x \mapsto \sqrt{x}$, which is Borel-measurable.
So $f^{-1}(E)$ is Borel, this is $\{x \in \mathbb{R}_{\geq 0} : \sqrt{x} \in E\}$.
If we take the $X^{-1}$ of this set, we get $E$. So $E \in \sigma(X)$.

PROBLEM 8.3b)

For arbitrary symmetric $E \in \mathcal{B}([-1,1])$, we must show

$$\int_E \hat{Y} dP = \int_E Y dP$$

Def. $j : x \mapsto -x$. The left side is equal to

$$\begin{aligned} \int_E \hat{Y} dP = \int_E \frac{1}{2}(Y(x) + Y \circ j(x)) dP(x) &= \frac{1}{2}\int_E Y(x) dP(x) + \frac{1}{2}\int_E Y \circ j(x) dP(x) \\ &= \frac{1}{2}\int_E Y(x) dP(x) + \frac{1}{2}\int_E Y(x) dP^j(x) \text{ (solution of problem 4.11)} \end{aligned}$$

Since $P^j(B) = P(j \in B) = P(j^{-1}(B)) = P(-B)$ and $P = \frac{1}{2}\lambda_{|\mathcal{B}[-1,1]}$, we get the equality as desired.

Since $Y \in L^1(\Omega, \mathcal{F}, P)$ and because what we have just shown, it now holds that

$$\int_{[-1,1]} \hat{Y} dP = \int_{[-1,1]} Y dP \in \mathbb{R}$$

So $\hat{Y}$ is also integrable. Finally, it is $\sigma(X)$-measurable, because

$$\hat{Y}(x) = Y(x) + Y(-x) = Y(-x) + Y(x) = \hat{Y}(-x)$$

PROBLEM 8.4)
Uniqueness has also been covered in the theory text, so we are left with existence.

We gonna show that there exist $\mathbb{E}(X|\mathcal{G})$ as defined, for $X \geq 0$. Once we have done that, we get for general $X$ that

$$\int_G X dP = \int_G X^+ dP - \int_G X^- dP = \int_G \mathbb{E}(X^+|\mathcal{G}) dP - \int_G \mathbb{E}(X^-|\mathcal{G}) dP$$
$$= \int_G \mathbb{E}(X^+|\mathcal{G}) - \mathbb{E}(X^-|\mathcal{G}) dP$$

so then $\mathbb{E}(X^+|\mathcal{G}) - \mathbb{E}(X^-|\mathcal{G})$ is an element in $L^1(\mathcal{G})$ which we wanna have.

From now on assume that $X$ as in this problem, is $\geq 0$.
This is the pointwise limit of simple functions, say $(X_k)_k$, and this is a sequence in $L^2$.
So it holds that

$$\int_G \mathbb{E}(X_k|\mathcal{G}) dP = \int_G X_k dP \xrightarrow{(MConvTh)} \int_G Y dP \tag{1}$$

Where we wanna go now, is saying $\int_G \mathbb{E}(X_k|\mathcal{G}) dP \to \int_G Y^* dP$ for some $Y^* \in L^1(\mathcal{G})$.

Since $(X_k)_k$ can be taken to be increasing, we get

$$0 \leq \mathbb{E}(X_{k+1} - X_k|\mathcal{G}) \text{ (cor. 8.4) } = \int_G X_{k+1} - X_k dP$$
$$= \int_G X_{k+1} dP - \int_G X_k dP = \int_G \mathbb{E}(X_{k+1}|\mathcal{G}) dP - \int_G \mathbb{E}(X_k|\mathcal{G}) dP$$

So $\left( \int_G \mathbb{E}(X_k|\mathcal{G}) dP \right)_k$ is increasing.

Further

$$\int |\mathbb{E}(X_m - X_n)|\mathcal{G})| dP \leq \int |X_m - X_n| dP \text{ (lemma 8.6) } \xrightarrow{m,n\to\infty} 0$$

So $\left( \int_G \mathbb{E}(X_k|\mathcal{G})dP \right)_k$ is a Cauchy-sequence in $L^1$.
Since $L^1$ is Banach, this sequence has a limit $Y^*$ in $L^1$, so

$$\int_G |\mathbb{E}(X_k|\mathcal{G}) - Y^*|dP \to 0$$

We claim that the pointwise limit of $\mathbb{E}(X_k|\mathcal{G})$ is $Y^*$.
This is because

$$\int_G |Y^* - L|dP \leq \int_G |Y^* - \mathbb{E}(X_k|\mathcal{G})| + |\mathbb{E}(X_k|\mathcal{G}) - L|dP \to 0$$

hence $\int_G |Y^* - L|dP = 0$ hence $|Y^* - L| = 0$ a.e. (lemma 4.11) hence $Y^* = L$ a.e.

Now we get

$$\int_G \mathbb{E}(X_k|\mathcal{G})dP \to \int_G Y^* dP \text{ (MconvT)}$$

so we are done by (1).

PROBLEM 8.9)
We must show $\hat{h}(Y)$ is an element of $L^1(\sigma(Y))$ such that if $B \in \mathcal{B}$ then

$$\int_{\{Y \in B\}} \hat{h}(Y)dP = \int_{\{Y \in B\}} h(X)dP$$

We have to assume $h$ is measurable. Consequence is that $\hat{h}$ is also measurable.
The left side is now meaningful and can be rewritten as follows:

$$\begin{aligned} \int_{\{Y \in B\}} \hat{h}(Y)dP &= \int 1_B(Y)\hat{h}(Y)dP = \int 1_B(y) \int h(x)\hat{f}(x|y)dx f_Y(y)dy \\ &= \int \int h(x)1_B(y)f(x,y)dxdy = \int h(x)1_B(y)f(x,y)dxdy \text{ (Fubini)} \\ &= \int h(X)1_B(Y)dP = \int_{\{Y \in B\}} h(X)dP \end{aligned}$$

So the equality holds, also if $\{Y \in B\} = \mathbb{R}$, in which case the right side is in $\mathbb{R}$ because $\mathbb{E}|h(X)| < \infty$, hence the left side is also in $\mathbb{R}$. So we now have that $\hat{h}(Y)$ is an element in $L^1(\sigma(Y))$ which satisfies what we wanted to have.